J. Fluid Mech. (1975), *vol.* 68, *part* 2, *pp.* 403–412
Printed in Great Britain

On the stability of shear flow in a rotating gas

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(Received 2 May 1974)

The linear stability of a rotating contained perfect gas flowing axially is shown to depend on a 'centrifugal Richardson number' when the ratio of the axial flow to the peripheral velocity, and the ratio of the peripheral velocity to the sound speed are both small. The Boussinesq approximation is not made. In the limit of infinite sound speed the known incompressible result (unstable, Pedley 1968) is reproduced. Comparison with computational results indicates that the asymptotic theory is pessimistic.

1. Introduction

The stability of shearing flow in a rotating gas is relevant to numerous geophysical and astrophysical problems concerning planetary and stellar atmospheres. These problems have provided a fruitful field for the application of the Boussinesq equations with numerous additional features, such as an independent stratification. Similar problems are also encountered in engineering applications such as the flow in compressors and turbines.

In this paper I have isolated a simple problem to which one can apply the full compressible equations of motion and obtain a linear stability criterion. I have chosen gas flow in a pipe which is characterized by an azimuthal flow which is almost a solid-body rotation and an axial flow small compared with the rotational velocity. The conclusions apply as well to a rotating annulus. This simple case is in the spirit of, among others, Howard & Gupta (1962), Pedley (1968, 1969) and Maslowe (1974).

The compressible problem is complicated by the difficulty of obtaining an exact solution around which to linearize. This problem will be avoided here by supposing from the start that the gas is strictly inviscid. This hypothesis is justified by noting that the inviscid stability criterion so obtained is less constraining for a slightly stratified fluid than the viscous criterion given by Pedley (1969), so that flows which are unstable according to the inviscid criterion cannot be stabilized by reasonable viscosities. This argument holds for Stewartson layers within the range of parameters considered in this paper. I shall discuss this point further in § 4.

The eigenvalue problem obtained by linearizing about such a steady motion and seeking solutions which behave like $e^{i\sigma t}$ appears to be beyond the scope of analysis. Some computer results exist (Plobeck 1974), however the essential physics are likely to be discovered analytically. Therefore I have studied an asymptotic problem for which both the ratio of the axial velocity to the rotational velocity, and the ratio of the rotational velocity to the sound speed are small. I have restricted the class of infinitesimal disturbances to those which have a small ratio of axial to azimuthal wavenumber. With the exception of the 'Mach number' criterion these are precisely the conditions imposed by Pedley (1968). The results obtained below contain Pedley's result as a special case.

No assumptions, other than the vanishing of viscosity, are made about the physical properties of the (perfect) gas. Because of the special character of adiabatic and isothermal states I have chosen to impose a one-parameter family of basic temperature states varying continuously between the two extremes. This arbitrariness is consistent with the arbitrariness introduced in the velocity profiles.

The primary result of this paper is that the flow is unstable to the disturbances considered if an asymptotic 'centrifugal Richardson number' is anywhere less than $\frac{1}{4}$. The unstable eigenfunctions appear to be confined to the regions in which this criterion is violated, but no attempt has been made to follow the development of the flow to finite amplitude. Only in the case of Poiseuille flow, for which this Richardson number is a constant, is it possible to give the eigenfunction explicitly.

The usual Richardson number can be written as

$$Ri = \mathbf{g} . \nabla\rho/\rho W'^2, \tag{1.1}$$

where $\mathbf{g}$ is the gravity vector, ρ the density and W' the derivative of the shear velocity in the direction of gravity. For the centrifugal compressible case, $\mathbf{g}$ can be replaced by the centripetal acceleration and $\rho^{-1}\rho'$ by

$$\left(\frac{1}{\rho}\frac{d\rho}{d\varpi}\right)_{\text{adia}} - \frac{1}{\rho}\frac{d\rho}{d\varpi}, \tag{1.2}$$

where ϖ is the dimensionless radial co-ordinate.

The asymptotic expression for this Richardson number may be written as

$$Ri_* = \Omega^2\varpi^2(1-\alpha)(\gamma-1)\mu^2/W'^2, \tag{1.3}$$

where Ω, α, γ, μ and W are the rate of basic rotation, the ratio of the temperature gradient in the gas to the adiabatic temperature gradient, the ratio of the specific heats, the ratio of the peripheral rotation rate to the sound speed at the wall and the local axial velocity, respectively. The prime denotes the radial derivative. As will become clear below, $Ri_* \leqslant Ri$ throughout the gas. Equality holds for the isothermal ($\alpha = 0$) case.

If one supposes that shearing rates are limited by Stewartson-layer thicknesses then one can derive the dependence of Ri_* on Ω for fixed axial velocities and for fixed mass fluxes:

$$Ri_*(W) \propto \Omega^{\frac{4}{3}}, \quad Ri_*(\dot{M}) \propto \Omega^{\frac{2}{3}}. \tag{1.4}$$

Finally it should be noted that the computer results indicate that actual flows are more stable than the asymptotic theory indicates.

The plan of the paper is as follows. Section 2 describes a general basic state and the associated perturbation equations. Section 3 is the heart of the paper. There the asymptotic stability criterion is derived. Section 4 compares the asymptotic results with some computer results for Poiseuille flow and gives a brief discussion.

2. Formulation

This investigation is concerned with flow in a pipe of radius a and infinite length rotating about its symmetry axis with angular velocity Ω. The gas is assumed to be a perfect gas and perfectly inviscid. Its motion is governed by conservation of momentum,

$$\rho' D\mathbf{V}'/Dt' + \nabla P' = 0, \tag{2.1}$$

of mass,

$$\partial\rho'/\partial t' + \nabla . (\rho'\mathbf{V}') = 0, \tag{2.2}$$

and of entropy,

$$DS'/Dt' = 0, \tag{2.3}$$

and an equation of state, which may be written symbolically in terms of pressure, density and entropy,

$$P' = P(\rho', S'), \tag{2.4}$$

as well as in the familiar perfect gas law form,

$$P' = R\rho' T'. \tag{2.5}$$

Replacement of the full equation of conservation of internal energy by (2.3) is consistent with the neglect of viscosity in (2.1). The symbols $\mathbf{V}'$, ρ', P', S', T', R and t' denote velocity, density, pressure, entropy, temperature, the gas constant and time, respectively.

It is convenient to define a cylindrical co-ordinate system (ϖ', Φ, z') in which $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$ and the wall of the pipe is $\varpi' = a$. The reader may verify that

$$\mathbf{V}'_0 = V'(\varpi')\hat{\mathbf{\Phi}} + W'(\varpi')\hat{\mathbf{z}} \tag{2.6}$$

and P'_0, ρ'_0, S'_0 and T'_0 functions of ϖ' alone will satisfy (2.1)–(2.5). One is at liberty to specify $V'(\varpi')$, $W'(\varpi')$ and one of the thermodynamic variables.

Two special thermodynamic conditions are of more than passing interest: adiabatic and isothermal. They can both be included in the model by defining the parameter α:

$$\frac{dT'_0}{d\varpi'} = \alpha\left(\frac{dT'_0}{d\varpi'}\right)_{\text{adia}}. \tag{2.7}$$

One can find the adiabatic temperature gradient by differentiating (2.4) and (2.5) with respect to ϖ' and setting the entropy gradient to zero. By substitution from (2.1) one finds that

$$\left(\frac{dT'_0}{d\varpi'}\right)_{\text{adia}} = \frac{\gamma-1}{\gamma R}\frac{V'^2(\varpi')}{\varpi'}, \tag{2.8}$$

where γ is the ratio of specific heats.

Combination of (2.7) and (2.8) suffices to prescribe the temperature profile, and one can write the basic solution, actually a one-parameter family of basic solutions, as

$$\left.\begin{aligned}
\mathbf{V}'_0(\varpi') &= V'(\varpi')\hat{\mathbf{\Phi}} + W'(\varpi')\hat{\mathbf{z}},\\
\rho'_0(\varpi') &= \rho_w C^{2q},\\
P'_0(\varpi') &= RT_w\rho_w C^{2(q+1)},\\
T'_0(\varpi') &= T_w - \frac{\alpha(\gamma-1)}{\gamma R}\int_{\varpi'}^{a}\frac{V'^2(s)}{s}\,ds,
\end{aligned}\right\} \tag{2.9}$$

where the subscript w denotes evaluation at the wall and

$$C^2 = 1 - \frac{\alpha(\gamma-1)}{\gamma R T_w}\int_{\varpi'}^{a} \frac{V'^2(s)}{s}\,ds, \quad q = \frac{\alpha+\gamma(1-\alpha)}{\alpha(\gamma-1)}. \tag{2.10}$$

In the isothermal ($\alpha \to 0$) limit, $q \to \infty$ and the density and pressure become exponential functions, viz.

$$\left.\begin{aligned} \rho_0' &\to \rho_w \exp\left\{-\int_{\varpi'}^{a} \frac{V'^2(s)}{RT_w s}\,ds\right\}, \\ P_0' &\to RT_w\rho_w \exp\left\{-\int_{\varpi'}^{a} \frac{V'^2(s)}{RT_w s}\,ds\right\}. \end{aligned}\right\} \tag{2.11}$$

It is convenient to non-dimensionalize the original system according to the scheme

$$\left.\begin{aligned} \mathbf{r}' = a\mathbf{r}, \qquad t' = \Omega^{-1}t, \quad \mathbf{V}' = \Omega a\mathbf{v}, \\ P' = \rho_w\Omega^2a^2P, \quad \rho' = \rho_w\rho_*, \quad T' = T_wT. \end{aligned}\right\} \tag{2.12}$$

The non-dimensional version of the basic solution is then

$$\left.\begin{aligned} \mathbf{v}_0 = V(\varpi)\hat{\mathbf{\Phi}} + W(\varpi)\hat{\mathbf{z}}, \\ \rho_0 = C^{2q}, \quad P_0 = (\gamma\mu^2)^{-1}C^{2(q+1)}, \quad T_0 = C^2, \end{aligned}\right\} \tag{2.13}$$

where $\mu^2 = \Omega^2a^2/\gamma RT_w$ is the square of a Mach number and

$$C^2 = 1 - \alpha(\gamma-1)\mu^2\int_{\varpi}^{1} \frac{V^2(r)}{r}\,dr \tag{2.14}$$

is the fully non-dimensional form of the quantity defined by (2.10).

Equations (2.1)–(2.3) are unchanged in form. The convective derivative of (2.4) is required below and can be written, using (2.3), as

$$(\partial/\partial t + \mathbf{v}.\nabla)P = C^2\mu^{-2}(\partial/\partial t + \mathbf{v}.\nabla)\rho_*. \tag{2.15}$$

Equation (2.5) is transformed to

$$P = (\gamma\mu^2)^{-1}\rho T. \tag{2.16}$$

To form the equations for linear stability the basic solution is perturbed according to

$$\left.\begin{aligned} \mathbf{v} = \mathbf{v}_0 + \mathbf{u}, \quad \rho_* = \rho_0 + \rho, \quad P = P_0 + \rho_0 Q, \\ T = T_0 + \Theta, \quad S = S_0 + s. \end{aligned}\right\} \tag{2.17}$$

The latter two quantities are decoupled from the first three and need not be considered further. They can be calculated *a posteriori* if desired.

Solutions for $\mathbf{u}$, ρ and Q proportional to

$$\exp i[\sigma t + m\Phi + kz] \tag{2.18}$$

will be sought. With this *Ansatz* the first-order (in $\mathbf{u}$, ρ and P) form of (2.15) becomes an algebraic equation which one can use to eliminate ρ in terms of u and Q:

$$i\Lambda(\varpi)\rho_0 Q + uP_0' = (C^2/\mu^2)[i\Lambda(\varpi)\rho + u\rho_0'], \tag{2.19}$$

where $\Lambda(\varpi) = \sigma + Vm/\varpi + Wk$ and a prime henceforth denotes $\partial/\partial\varpi$. After some algebra one arrives at

$$\rho = -(1-\alpha)(\gamma-1)\frac{\mu^2}{C^2}\frac{V^2}{\varpi}\frac{\rho_0}{i\Lambda}u + \frac{\mu^2}{C^2}\rho_0 Q. \tag{2.20}$$

Combining (2.1), (2.2), (2.13), (2.17) and (2.20) leads to the four governing equations for this problem, viz.

$$i\Lambda\mathbf{u} - 2\frac{V}{\varpi}v\hat{\boldsymbol{\varpi}} + \left(\frac{V}{\varpi} + V'\right)u\hat{\boldsymbol{\Phi}} + Wu\hat{\mathbf{z}} + (\gamma-1)\frac{V^2}{\varpi}\frac{\mu^2}{C^2}Q\hat{\boldsymbol{\varpi}}$$
$$+ (1-\alpha)(\gamma-1)\frac{\mu^2}{C^2}V^2\frac{1}{i\Lambda}u\hat{\boldsymbol{\varpi}} + \nabla Q = 0, \tag{2.21a}$$

$$i\Lambda\frac{\mu^2}{C^2}Q + \frac{\mu^2}{C^2}\frac{V^2}{\varpi}u + \nabla.\mathbf{u} = 0. \tag{2.21b}$$

The boundary conditions for this set are that $u(1) = 0$ and that everything be suitably well behaved at the origin.

Below it will be necessary to restrict $V(\varpi)$:

$$V(\varpi) = \varpi + \delta v(\varpi). \tag{2.22}$$

This makes $$\Lambda(\varpi) = \sigma + m + \frac{\delta v}{\varpi}m + Wk = \lambda + \frac{\delta v}{\varpi}m + Wk. \tag{2.23}$$

3. Asymptotic stability

The eigenvalue problem represented by (2.21), (2.22) and the boundary conditions seems analytically intractable. Plobeck (1974) has obtained some numerical results using Poiseuille flow with $\delta v \equiv 0$ by eliminating everything but the radial velocity from the set (2.21) and 'shooting': numerically integrating the resulting second-order ordinary differential equation for different values of σ until the boundary conditions are satisfied. Analytic results can be obtained asymptotically.

I introduce a small parameter ϵ in order to make the asymptotic analysis easier to follow. The quantities required to be small are then supposed to be proportional to ϵ, viz.

$$\mu = \mu_*\epsilon, \quad k = k_* m\epsilon, \quad W = W_*\epsilon, \quad \delta v = \delta v_*\epsilon^2, \tag{3.1}$$

where the starred quantities are $O(1)$. The first derivative of W_* must also be $O(1)$.

Having made the substitution (3.1) one supposes that the solution $(\mathbf{u}, Q)$ and the eigenvalue $\lambda = \sigma + m$ are expandable in power of ϵ:

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \ldots, \\ Q &= Q_0 + \epsilon Q_1 + \ldots, \\ \lambda &= \epsilon\lambda_1 + \ldots. \end{aligned}\right\} \tag{3.2}$$

Substitution of (3.1) and (3.2) then leads to a series of problems at different orders in ϵ. (It should be remarked in passing that this procedure is equivalent to reducing (2.1) to a single equation for Q, supposing $\lambda = \lambda_*\epsilon$ and making the hypothesis (3.1): the procedure followed by Pedley (1968). I have used this alternative procedure because I find it clearer.)

The first such problem is

$$\left.\begin{aligned} -2v_0+Q_0' &= 0,\\ 2u_0+(im/\varpi)\,Q_0 &= 0,\\ (\varpi u_0)'+imv_0 &= 0,\\ u_0(1) &= 0.\end{aligned}\right\} \tag{3.3}$$

It has an infinity of solutions:

$$u_0 = (-im/2\varpi)\,Q_0, \quad v_0 = \tfrac{1}{2}Q_0', \tag{3.4}$$

subject only to the conditions that $Q_0(1) = 0$ and that Q_0 and Q_0' be sufficiently well behaved at the origin.

The $O(\epsilon)$ equations lead to a more interesting result. These are written as

$$\left.\begin{aligned} \left[i\lambda_1+\frac{(1-\alpha)(\gamma-1)\mu_*^2}{i\lambda_1}\varpi^2\right]u_0-2v_1+Q_1' &= 0,\\ 2u_1+i\lambda_1v_0+(im/\varpi)\,Q_1 &= 0,\\ W_*'u_0+i\lambda_1w_0+ik_*mQ_0 &= 0,\\ (\varpi u_1)'+imv_1+ik_*m\varpi w_0 &= 0.\end{aligned}\right\} \tag{3.5}$$

These equations may be reduced by substituting for u_0 and v_0 from (3.4) and then eliminating u_1, v_1 and w_0 in favour of Q_0 and Q_1 using the first three equations. When this result is substituted into the last equation, the terms involving Q_1 are identically zero and one is left with a single equation for Q_0, viz.

$$(\varpi Q_0')'+\left[\frac{1}{\lambda_*^2}g(\varpi)-\frac{m^2}{\varpi}\right]Q_0 = 0, \tag{3.6}$$

subject to the boundedness condition at the origin and $Q_0(1) = 0$. The function $g(\varpi)$, which controls the behaviour of the solutions to (3.6), is given by

$$g(\varpi) = 2k_*m^2(2k_*\varpi-W_*')+m^2(\gamma-1)(1-\alpha)\mu_*^2\varpi. \tag{3.7}$$

Equation (3.6) and its boundary conditions form a classical Sturm–Liouville problem for the eigenvalue $\xi = \lambda_*^{-2}$, and the following may be inferred from classical Sturm–Liouville theory (cf. Ince 1956, §10.61).

(i) If $g(\varpi) > 0$ throughout the interval, all the eigenvalues ξ_n are positive, the corresponding λ_n are real and the system is stable.

(ii) If $g(\varpi) < 0$ throughout the interval all the eigenvalues ξ_n are negative, the corresponding λ_n are imaginary and the system is unstable.

(iii) If $g(\varpi)$ changes sign in the interval there are two sets of eigenvalues $\{\xi_n\}^+$ and $\{\xi_n\}^-$. One of these contains positive and the other negative eigenvalues; there are both real and imaginary λ_n and the system is *unstable*.

From these it is clear that a necessary and sufficient condition for stability is that $g(\varpi) > 0$ throughout the region. This can be written as

$$|W_*'| < \{2|k_*|+(1-\alpha)(\gamma-1)\mu_*^2/2|k_*|\}\varpi. \tag{3.8}$$

The second term on the right-hand side of (3.8) is new; the first term is Pedley's incompressible result. The appearance of this second term means that the right-hand side has a minimum at

$$k_* = \tfrac{1}{2}\mu_*(\gamma-1)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}, \tag{3.9}$$

so that asymptotic stability is assured if

$$|W'_*| < 2(\gamma-1)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\mu_*\varpi. \tag{3.10}$$

That (3.10) represents the centrifugal Richardson number criterion asymptotically is seen by squaring (3.10), rearranging and redimensionalizing to obtain

$$\tfrac{1}{4} < (\gamma-1)(1-\alpha)\,\Omega^2\varpi^2\mu^2/W'^2 = Ri_*. \tag{3.11}$$

Since
$$\frac{1}{\rho_0}\frac{d\rho_0}{d\varpi} = \frac{\alpha(\gamma-1)\mu^2\Omega^2\varpi}{C^2} \tag{3.12}$$

and $C^2 \sim 1$, criterion (3.10) involves the centrifugal Richardson number defined in § 1 consistent with the asymptotic approximation. One notes also that Ri_* vanishes when $\alpha = 1$ (an adiabatic lapse rate). Because $C^2 \leqslant 1$ throughout the gas $Ri_* \leqslant Ri$.

Except for the case of Poiseuille flow, discussed below, the eigenfunctions cannot be found by the methods used above. However one can say something about their properties. If (3.6) is multiplied by $\bar{Q}_0$, the complex conjugate of Q_0, and integrated between zero and unity, one can obtain an expression for λ_*^2:

$$\frac{\lambda_*^2}{m^2} = \frac{\langle[4k_*^2+(\gamma-1)(1-\alpha)\mu_*^2]\,\varpi Q_0\bar{Q}_0\rangle - 2k_*\langle W'_*Q_0\bar{Q}_0\rangle}{\langle\varpi Q'_0\bar{Q}'_0 l\rangle + m^2\langle\varpi^{-1}Q_0\bar{Q}_0\rangle}. \tag{3.13}$$

In this expression the angular brackets denote the integration. The reader will note that every term but the second in the numerator is positive definite. Sturm–Liouville theory ensures that $\lambda_*^2 < 0$ for sufficiently large $|W'_*|$; equation (3.12) shows that the eigenfunctions must be concentrated in the region(s) of maximum shear.

Poiseuille flow is a special case: Ri_* is constant. If

$$W = \mathscr{W}(1-\varpi^2) \tag{3.14}$$

then the criterion (3.10) reduces to

$$\mathscr{W} < (\gamma-1)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\mu. \tag{3.15}$$

The function $g(\varpi)$ is then a linear function of ϖ and (3.6) reduces to Bessel's equation, so that

$$Q_0 = J_m\left\{\frac{[4k_*m^2(k_*-\mathscr{W})+m^2(\gamma-1)(1-\alpha)\mu_*^2]^{\frac{1}{2}}}{\lambda_*}\varpi\right\} \tag{3.16}$$

and
$$\lambda_{*n} = [4k_*m^2(k_*-\mathscr{W})+m^2(\gamma-1)(1-\alpha)\mu_*^2]^{\frac{1}{2}}/j_{mn}, \tag{3.17}$$

where j_{mn} is the nth zero of the mth-order Bessel function.

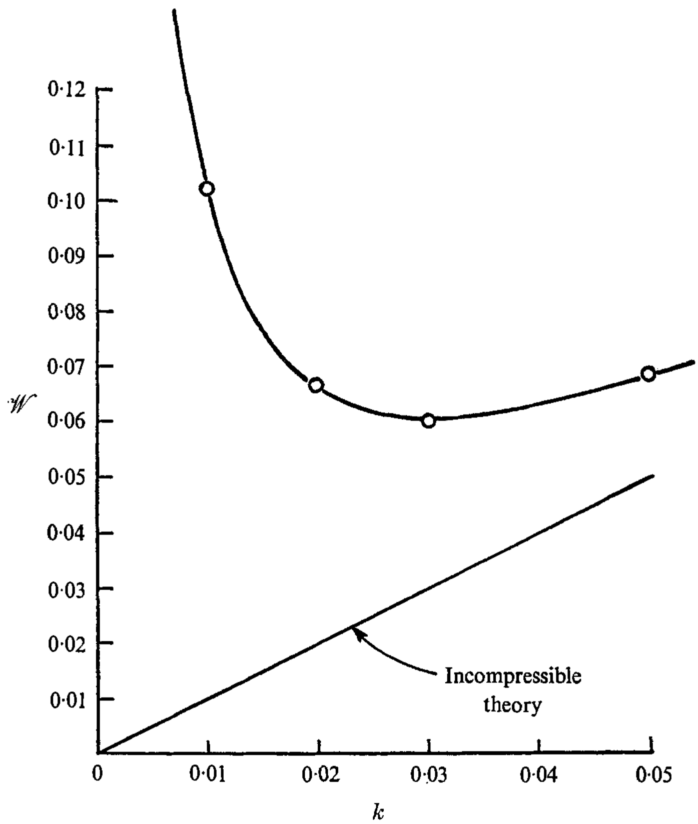

FIGURE 1. Comparison of computed critical velocities for isothermal Poiseuille flow with the asymptotic theory: $\gamma = 1{\cdot}09$, $m = -1$, $\mu = 0{\cdot}2$.

4. Discussion

It is informative to compare an asymptotic solution such as that developed above with real solutions beyond the asymptotic range. Unfortunately I have been unable to uncover any applicable experimental results and can only compare these results with the numerical results of Plobeck (1974), valid for Poiseuille flow. Figures 1 and 2 show the theoretical curve $\mathscr{W}$ *vs.* k obtained by replacing the inequality in (3.8) by equality and replacing W by the form for Poiseuille flow defined by (3.14). The points on figure 1 were found by L. Hultgren using Plobeck's computer program; those on figures 2 (*a*) and (*b*) are taken from Plobeck (1974).

Figure 1 is truly asymptotic, and the agreement is as it should be. Figures 2 (*a*) and (*b*) are beyond the asymptotic range. One notes that the general shape of the curve is reproduced by the data points, and that the computer calculations indicate that the actual flow is more stable than the asymptotic analysis suggests.

The question of the realizability of the basic flow has been left somewhat open. Poiseuille flow is the only simple realizable flow meeting the restrictions set on δv. However, if the viscosity is sufficiently small and the pipe sufficiently long (to suppress z dependence and radial motions), then an axial velocity $O(\epsilon)$ will induce, by nonlinear interaction, a δv which is, as required, $O(\epsilon^2)$.

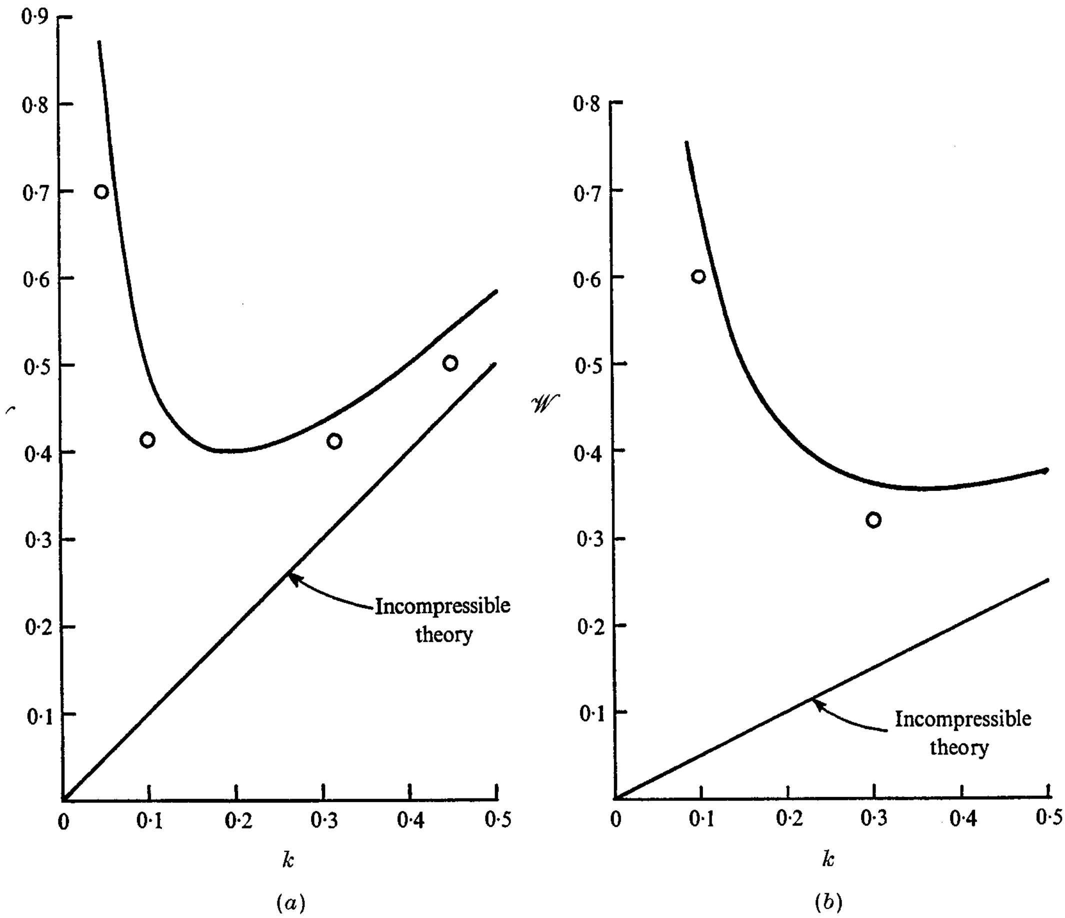

FIGURE 2. Stability beyond the asymptotic range: $\gamma = 1{\cdot}09$, $\mu = 1$. (*a*) $m = -1$. (*b*) $m = -2$. Data from Plobeck (1974).

More can be said about the radial dependence of W. When $\mu \ll 1$ the scale height will be large compared with local measures of free shear layers and one can suppose that these will be $O(E^{\frac{1}{3}})$, where E is the Ekman number, so that $W' \approx O(WE^{-\frac{1}{3}})$ or smaller. Thus, if $\overline{W}$ is a characteristic dimensionless axial velocity, (3.10) can be amended to read

$$\overline{W} < 2(\gamma-1)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\mu\varpi E^{\frac{1}{3}}. \tag{4.1}$$

Under these circumstances the Reynolds number criterion given by Pedley (1969) can be expected to hold. His criterion is that the flow is stable if the Reynolds number is less than 82·9. In the notation of this paper, in a shear layer,

$$(\Omega L)\,\overline{W}(E^{\frac{1}{3}}L)/\nu < 82{\cdot}9, \tag{4.2}$$

or

$$\overline{W} < 82{\cdot}9E^{\frac{2}{3}}. \tag{4.3}$$

The ratio of maximum permissible velocities according to the two criteria is then

$$82{\cdot}9E^{\frac{1}{3}}/2(\gamma-1)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\mu\varpi \propto \Omega^{-\frac{1}{3}}\nu^{\frac{1}{3}}. \tag{4.4}$$

For sufficiently small viscosity the stability is truly determined by the stratification.

The fact that the stability criterion deduced above is a Richardson number criterion leads one to ask two questions. First, is the criterion correct away from its asymptotic region? Second, is this also an absolute stability bound in that no subcritical stabilities are possible? Comparison of my results with the numerical integrations of the full linear stability equation (Plobeck 1974) indicates that the answer to the first question remains open. I have spent a little time considering the question of absolute stability and have so far been unable to find even as much as an energy theorem of the sort found by Serrin (1959).

I thank L. Hultgren for figure 1. Much of the work was done while I was at M.I.T. and I thank L. N. Howard, M. T. Landahl and W. V. R. Malkus for helpful comments.

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